Nonlinear micro income processes with macro shocks

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November 12, 2024

Motivation

- How much are households willing to give up to eliminate business cycles?
  - Lucas (1987, 2003): 0.05% of consumption.
  - Representative agents with frictions and heterogeneous agents: [0.01%, 7.4%].
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- Typical answer relies on structural models:
  - Calibration/estimation using macro time series/micro data moments.
  - Discretization + log-linearization + linear state-space techniques.
  - Recent progress in estimation using micro data directly, but not yet widely adopted.

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  - Recent progress in estimation using micro data directly, but not yet widely adopted.
- Goal: nonlinear reduced form with rich micro dynamics and macro uncertainty.

This paper

- Nonlinear micro income process with macro business cycle state:

$$\eta_{it} = Q_\eta(\eta_{i,t-1}, Z_t, Z_{t-1}, u_{it}), \qquad Z_t = Q_Z(Z_{t-1}, V_t),$$

u_{it} and V_t are micro and macro shocks, η_{it} and Z_t are potentially unobserved.

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- We study the (nonparametric) identification of the model:
  - Time series of panels + macroeconometric/microeconometric ID techniques.
- We propose estimation and inference using a flexible parametric version:
  - Can be implemented via stable simulation-based algorithms.
  - Robust to some forms of cross-sectional dependence in u_{it} (omitted macro shocks).

This paper

- We develop methodology for impulse response function analysis:
  - What does "shock" mean in our reduced-form/semi-structural setup?
  - How should we measure the importance of macro/micro shocks?
  - Define local shocks that match a certain state perturbation experiment.

This paper

- We develop methodology for impulse response function analysis:
  - What does "shock" mean in our reduced-form/semi-structural setup?
  - How should we measure the importance of macro/micro shocks?
  - Define local shocks that match a certain state perturbation experiment.
- We fit our model to US household income + macro time series data:
  - Time series of panels spanning 1970-2019 from the PSID (seven recessions).
- Main findings:
  - Income persistence increases for low-η and decreases for high-η during recessions.
  - Recessions shift conditional skewness of η towards negative at most income levels.
  - Large cost of business cycles from nonlinear micro impact of macro shocks.

Selected literature

- Income dynamics (with and without business cycles):
  Storesletten, Telmer, Yaron (2004), Guvenen, Ozkan, Song (2014), Arellano, Blundell, Bonhomme (2017), Guvenen, McKay, Ryan (2022), Guvenen, Pistaferri, Violante (2022), GRID project, ...
- Heterogeneous agents estimation using micro data:
  Arellano, Bonhomme (2017), Liu, Plagborg-Møller (2023), Chang, Chen, Schorfheide (2024)
- Nonlinear IRFs:
  Gallant, Rossi, Tauchen (1993)

➡ This paper: semi-structural + nonlinear micro dynamics + aggregate shocks

Model and identification

Model: income process

- Permanent+transitory decomposition of (log) income y:

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Model: income process

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- Nonlinear income process with macro state variable Z_t:

$$\eta_{it} = Q_\eta(\eta_{i,t-1}, Z_t, Z_{t-1}, u_{it}),$$
$$\varepsilon_{it} = Q_\varepsilon(Z_t, Z_{t-1}, e_{it}),$$

u_{it}, e_{it} mutually/serially independent $U(0,1)$ rv's conditional on $\{Z_\tau\}$.

- Could also include observable covariates x_{it} (e.g., age) in Q_η, Q_ε.

Model: business cycle state

- Factor model for macro data $W = (\text{GDP, C, I, urate, hours})$:

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$$Z_t = \Phi Z_{t-1} + \Sigma_V^{1/2} V_t = Q_Z(Z_{t-1}, V_t),$$
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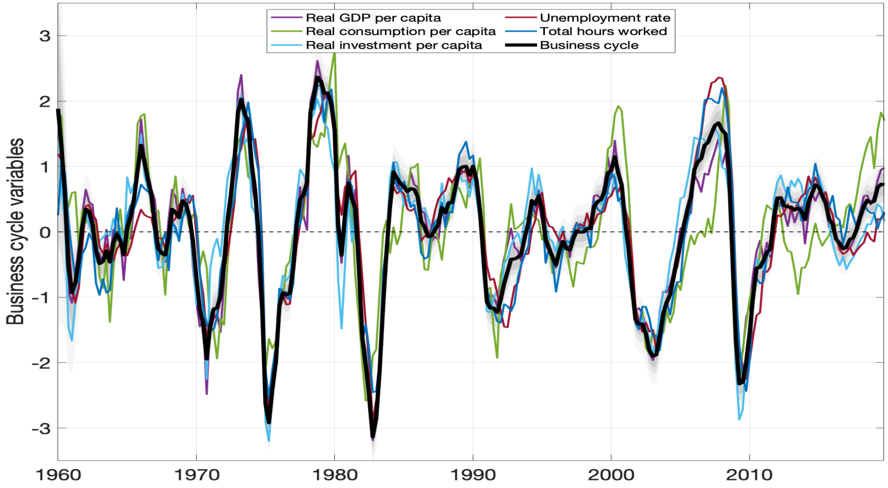
  V_t, e_t mutually/serially independent $N(0, I)$ rv's.

- Could allow for higher-order AR dynamics, heteroskedasticity, nonlinearities, etc.

Model: macro state in US data



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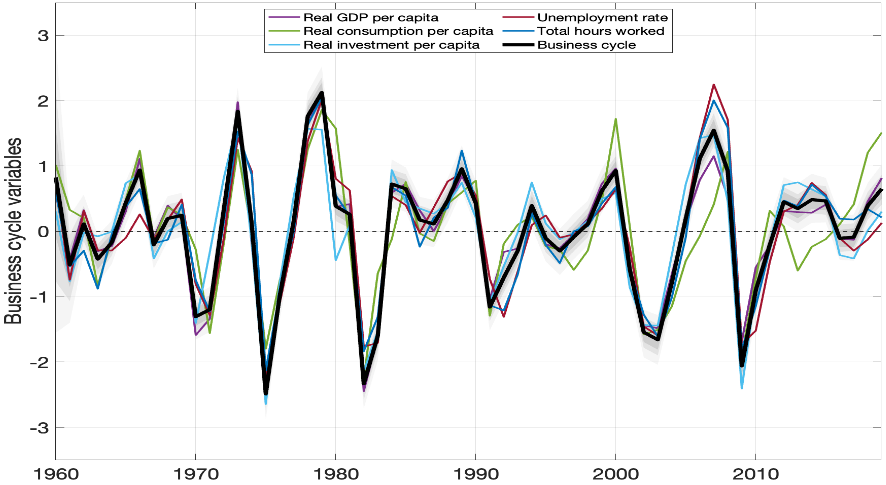

Objects of empirical interest

- Measures of nonlinear persistence and skewness:

$$\rho(u, \eta, \tilde{Z}, Z) = \frac{\partial Q_\eta(\eta, \tilde{Z}, Z, u)}{\partial \eta},$$

$$\text{sk}(\eta, \tilde{Z}, Z) = \frac{Q_\eta(\eta, \tilde{Z}, Z, u_0) + Q_\eta(\eta, \tilde{Z}, Z, 1 - u_0) - 2Q_\eta\left(\eta, \tilde{Z}, Z, \frac{1}{2}\right)}{Q_\eta(\eta, \tilde{Z}, Z, u_0) - Q_\eta(\eta, \tilde{Z}, Z, 1 - u_0)}.$$

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➡ Identification, estimation, inference of (functionals of) Q_η, Q_Z.

Aside: reduced forms for heterogeneous agents models

- Recursive equilibria typically imply Markovian law of motion (Q_η, Q_Z).
- Example.
  - Consumer maximizes $E\left[\sum_{t=1}^{\infty} \beta^t u(c_{it})\right]$ subject to sequence of budget equations $b(c_{it}, \eta_{it} + \varepsilon_{it}, a_{i,t+1}, a_{it}, R_t) = 0$ with assets a_{it} and interest rate R_t.
  - Under bounded rationality (à la Krusell, Smith (1998)),

$$
\begin{aligned}
c_{it} &= g_c(\eta_{it}, \varepsilon_{it}, a_{it}, \bar{\mathbf{Z}}_t), \\
a_{i,t+1} &= g_a(\eta_{it}, \varepsilon_{it}, a_{it}, \bar{\mathbf{Z}}_t), \\
\bar{\mathbf{Z}}_t &= (Z_t, R_t, \text{moments of } a_{it}\text{-distribution})
\end{aligned}
$$

  - $\bar{\eta}_{it} = (\eta_{it}, \varepsilon_{it}, c_{it}, a_{it})$ and $\bar{\mathbf{Z}}_t$ follow restricted/multivariate version of (Q_η, Q_Z).

Identification: time series of panels

- Researcher knows densities $f\left(\{W_t\}_{t=-\infty}^{\infty}\right)$ and $f\left(\{y_{i,t+s-1}\}_{s=1}^{S} \middle| \{W_{t+s-1}\}_{s=1}^{S}\right)$.

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- Conceptually, infinite-sample counterpart to a researcher who observes
  - Time series of macro data W_t for $t = 1, \ldots, T$.
  - Time series of panels $\left\{\{y_{i,t+s-1}\}_{s=1}^{S}\right\}_{i \in \mathcal{I}_t}$ for $t = 1, \ldots, T$.
    - Note: $i \in \mathcal{I}_t$ does not preclude (nor imply) $i \in \mathcal{I}_\tau$ for $\tau \neq t$.
    - Special cases: rotating/overlapping panels, a single long panel.
    - Practical considerations: representativeness and colinear covariates (e.g., age).
    - Precedent: Storesletten, Telmer, Yaron (2004).
- Identification for large T and $N_t \equiv \#\mathcal{I}_t$ with S fixed.

Identification: main result

Identification of macro state process

Suppose (upper block of Λ) $= I$, eigenvals of Φ within the unit circle and Σ_E diagonal.
Then, Q_Z is identified from $f\left(\{W_t\}_{t=-\infty}^{\infty}\right)$.

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Identification of micro process with macro states

Suppose

1. $\{y_{i,t+s-1}\}_{s=1}^{S}$ independent of $\{W_{t+s-1}\}_{s=1}^{S}$ given $\{Z_{t+s-1}\}_{s=1}^{S}$,
2. $\left\{f(\{Z_{t+s-1}\}_{s=1}^{S} | \{W_{t+s-1}\}_{s=1}^{S} = \mathbf{w}) : \mathbf{w} \in \mathbb{R}^{\dim(W)S}\right\}$ complete,
3. $S \geq 4$ + boundedness/completeness on $f\left(\{y_{i,t+s-1}, \eta_{i,t+s-1}\}_{s=1}^{S} \middle| \{Z_{t+s-1}\}_{s=1}^{S}\right)$.

Then, Q_η is identified from $f\left(\{y_{i,t+s-1}\}_{s=1}^{S} \middle| \{W_{t+s-1}\}_{s=1}^{S}\right)$.

Identification: discussion

- Sketch of the proof:
  - (i) is similar to Liu, Plagborg-Møller (2023, A1.2) and implies

$$f\left(\mathbf{y}_i|\mathbf{w}\right) = \int f\left(\mathbf{y}_i|\mathbf{z}\right) f\left(\mathbf{w}|\mathbf{z}\right) d\mathbf{z}.$$

  - Operator $[L_{a|b}h](a) = \int f(a|b)h(b)db \implies L_{\mathbf{y}_i|\mathbf{z}} = L_{\mathbf{y}_i|\mathbf{w}} L_{\mathbf{w}|\mathbf{z}}^{-1}$ (inverse exists by (ii)).
  - (iii) delivers Arellano, Blundell, Bonhomme (2017) microeconometric identification.

- Key: concurrent time series variation between micro and macro data.

Data: household income

- PSID:
  - Interviewed an initial sample representative of US households in 1968.
  - Thereafter, kept track of initial households and their offspring:
    - E.g., if daughter moves out to form her own household, added as a new unit.
  - Refresher/immigrant samples to (try to) preserve representativity.
  - Interviews are annual between 1968 to 1997, biennial between 1999 to 2019.

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  - Interviews are annual between 1968 to 1997, biennial between 1999 to 2019.
- We use the PSID to construct a time series of panels:
  - Each panel has $S = 4$ and made biennial for comparability (but we use all years).
    - Note: y is biennial while Z is quarterly.
  - We look at male earnings and family income:
    - Male earnings: labor income of representative person (male).
    - Family income: labor income of representative person (male/married) and spouse + transfers.
  - $y =$ log income net of education/race/family size/state of residence/etc.

Estimation and inference

Estimation: flexible parametric model

- Flexible model of quantile functions:

$$
\begin{aligned}
Q_\eta(\eta, Z_t, Z_{t-1}, u) &= \sum_{j=1}^{J} \sum_{k=1}^{K} \psi_j(\eta)\theta_{jk}(u)\varphi_k(Z_t, Z_{t-1}) \\
&= \psi(\eta)'\Theta(u)\varphi(Z_t, Z_{t-1}) \\
&= \psi(\eta)'\Delta_t(u).
\end{aligned}
$$

  - $\psi(\cdot), \varphi(\cdot,\cdot)$ are vectors of known basis functions (e.g., orthogonal polynomials).
  - $\Theta(\cdot)$ is matrix of linear splines supported on $(\bar{u}_1, \ldots, \bar{u}_L) \implies$ denote parameters by θ.
  - $\Delta_t(\cdot)$ too but depends on time series of parameters δ_t.

- Nonparametric (Sieve) if we let $J, K, L \to \infty$, but we take a parametric perspective.

Estimation: complete data moments

- Similar approach to quantile functions of base-period η and ε.
- Once we know θ, we can recover persistence, skewness, IRFs, etc.

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- Once we know θ, we can recover persistence, skewness, IRFs, etc.
- If η observed, we could use linear quantile regressions: for $\tau = \bar{u}_1, \ldots, \bar{u}_L$,

$$E\left[\left(\psi(\eta_{i,t-1}) \otimes \varphi(Z_t, Z_{t-1})\right) \cdot \nu_\tau\left(\eta_{it} - \psi(\eta_{i,t-1})'\Theta(\tau)\varphi(Z_t, Z_{t-1})\right)\right] = 0_{d\times 1}$$

  where $\nu_\tau(\cdot)$ is the derivative of the "check" function $x \mapsto (\tau - 1[x < 0])x$.

- Our model specifies the distribution of (η, Z) given (y, W):
  - Pseudo-likelihood EM: Arcidiacono, Jones (2003), Arellano, Bonhomme (2016).

Estimation: stochastic EM algorithm

- Suppose $\{Z_t\}_{1\leq t\leq T}$ observed (e.g., smoothed estimates).
  - Notation: $\boldsymbol{\eta}_t = \{\eta_{i,t+s-1}\}_{1\leq s\leq S,\, i\in\mathcal{I}_t}$ and $\boldsymbol{y}_t = \{y_{i,t+s-1}\}_{1\leq s\leq S,\, i\in\mathcal{I}_t}$.

Algorithm. Start with $\hat{\theta}^{(0)}$ and for $m = 1, \ldots, M$:

1. Stochastic E step:
   - Draw $\{\boldsymbol{\eta}_t^{(m)}\}_{1\leq t\leq T}$ from $f\left(\{\boldsymbol{\eta}_t\}_{1\leq t\leq T} \middle| \{\boldsymbol{y}_t, Z_t\}_{1\leq t\leq T}, \hat{\theta}^{(m-1)}\right)$.
2. Pseudo M step:
   - Update $\hat{\theta}^{(m)}$ by linear quantile regressions using $\{\boldsymbol{\eta}_t^{(m)}, Z_t\}_{1\leq t\leq T}$.

Once finished, set $\hat{\theta} = (\mu M)^{-1} \sum_{m=(1-\mu)M}^{M} \hat{\theta}^{(m)}$ for some $0 < \mu < 1$.

Estimation: stochastic EM algorithm

- Suppose macro model parameter estimates $\hat{\lambda}$ estimated with macro data.
  - Notation: $\boldsymbol{\eta}_t = \{\eta_{i,t+s-1}\}_{1\leq s\leq S,\, i\in\mathcal{I}_t}$ and $\boldsymbol{y}_t = \{y_{i,t+s-1}\}_{1\leq s\leq S,\, i\in\mathcal{I}_t}$.

Algorithm. Start with $\hat{\theta}^{(0)}$ and for $m = 1, \ldots, M$:

1. Stochastic E step:
   - Draw $\{Z_t^{(m)}\}_{1\leq t\leq T}$ from $f(\{Z_t\}_{1\leq t\leq T} | \{W_t\}_{1\leq t\leq T}, \hat{\lambda})$.
   - Draw $\{\boldsymbol{\eta}_t^{(m)}\}_{1\leq t\leq T}$ from $f\left(\{\boldsymbol{\eta}_t\}_{1\leq t\leq T} \middle| \{\boldsymbol{y}_t, Z_t^{(m)}\}_{1\leq t\leq T}, \hat{\theta}^{(m-1)}\right)$.
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Sampling properties

- One concern is cross-sectional dependence in u_{it}. Factor model for ranks:

$$u_{it} = \Phi\left(\sqrt{1-\gamma^2}U_{it} + \gamma F_t\right),$$

U_{it} iid over i, t, F_t iid over t, mutually independent $N(0,1)$ conditional on $\{Z_\tau\}$.

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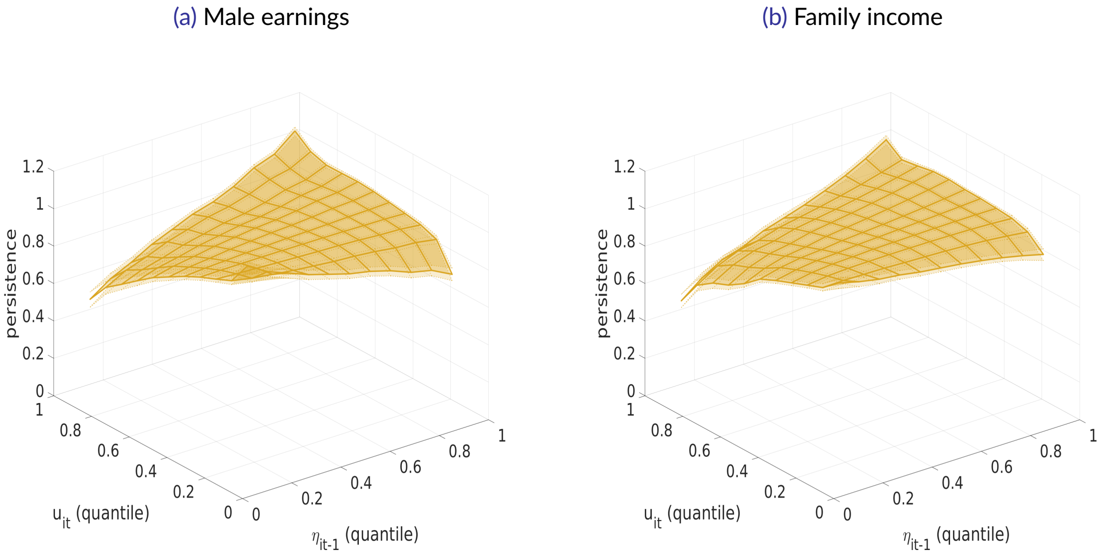
- Asymptotic properties. As $N, T \to \infty$, under regularity conditions,
  - Consistency: $\hat{\theta} \xrightarrow{\text{p}} \theta$.
  - Asymptotic normality: $\sqrt{T}(\hat{\theta} - \theta) \xrightarrow{\text{d}} N(0_{d\times 1}, \Omega)$.
    - Convergence rate is slower than standard panel rate $\sqrt{NT}$ unless $\gamma = 0$.

Empirical analysis: implementation

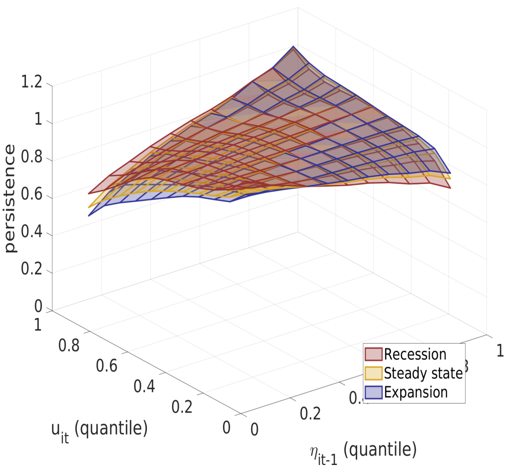
- Specification of Q_η:
  - ψ is third-order Hermite polyn on η $\times$ second-order Hermite polyn on age h.
  - φ is second-order Hermite polyn on (Z_t, Z_{t-1}) but with restrictions.
    - Linear term excluded from η, h interactions, quadratic term only enters intercept.
  - Grid on rank space has $L = 11$ and we model tails $u < \bar{u}_1$, $u > \bar{u}_L$ as exponential.
    - Q_η has $288 = 26 \times 11 + 2$ parameters, observation count $\approx$ 200.000.
- Specification of Q_{init} (base-period η) and Q_ε:
  - We include second-order Hermite polyn on age + $L = 11$ + exponential tails.
  - We use time effects instead of functions of (Z_t, Z_{t-1}).
- Confidence intervals via parametric bootstrap using factor model for ranks.

Nonlinear persistence in steady state

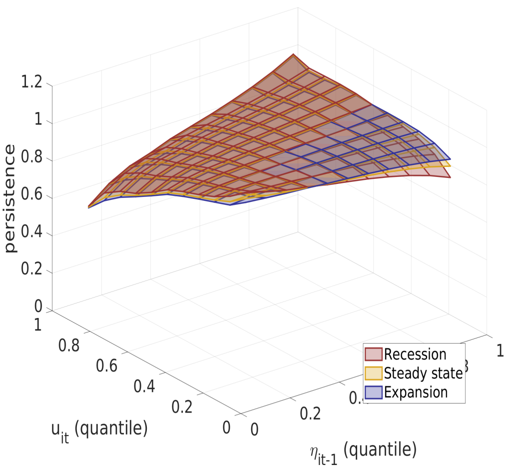


Nonlinear persistence over the business cycle

(a) Male earnings



(b) Family income



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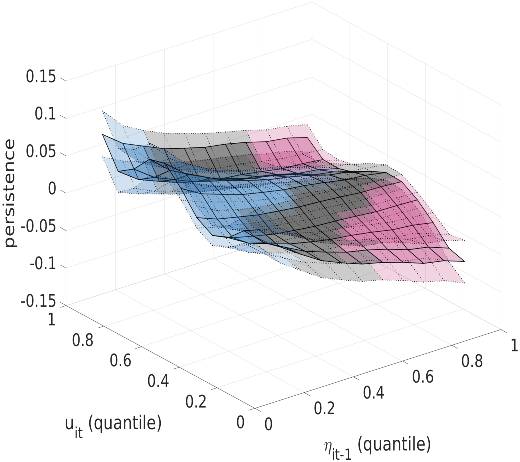

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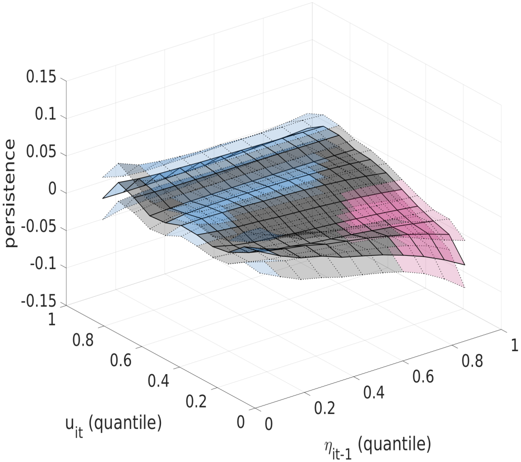

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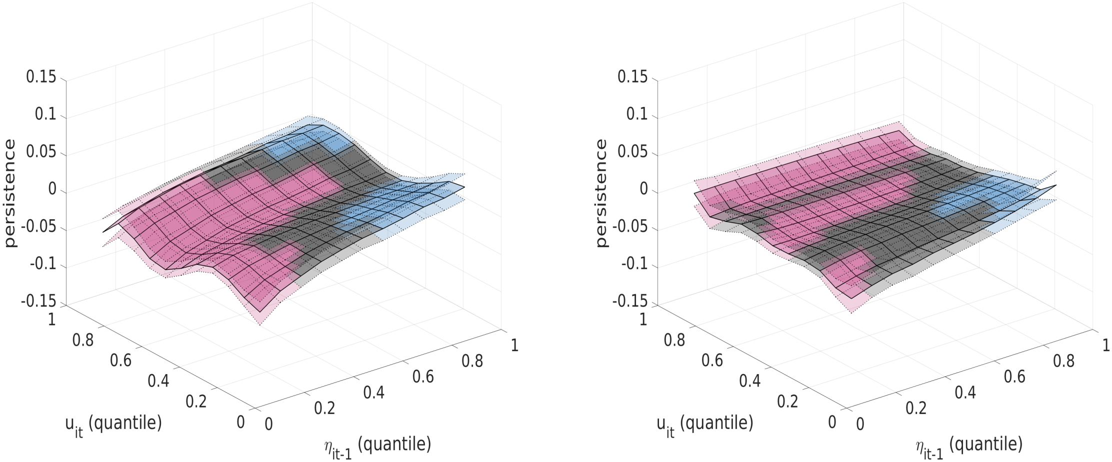

Conditional skewness over the business cycle

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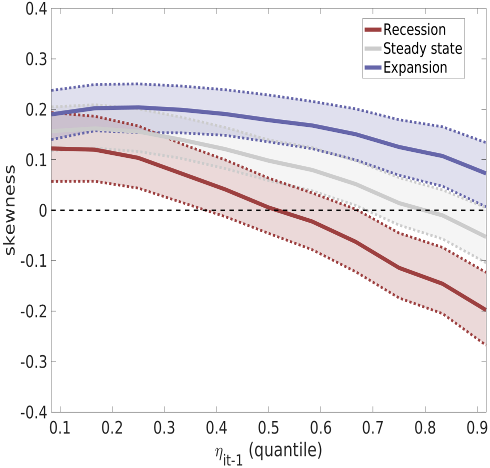

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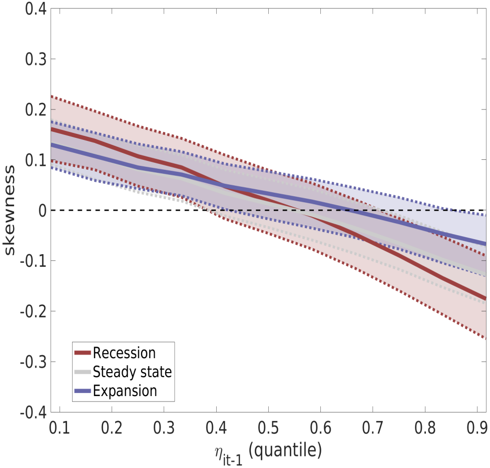

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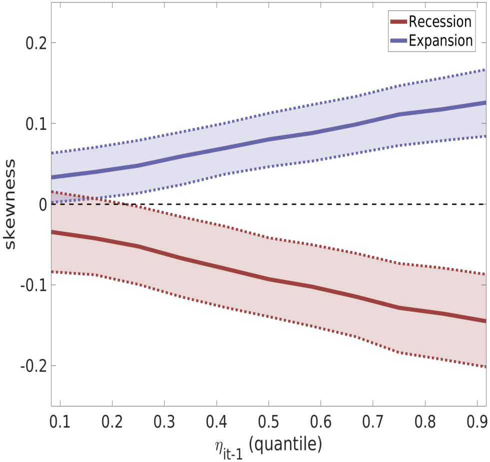

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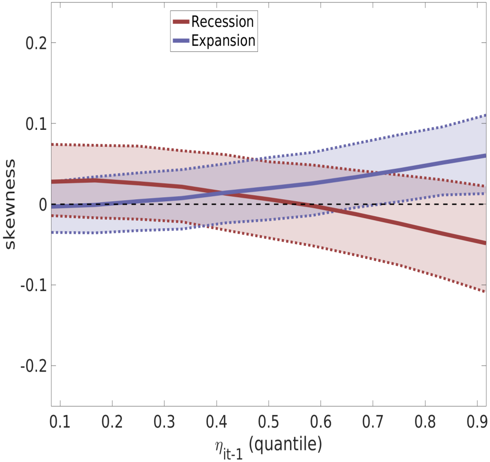

Impulse responses

Nonlinear IRFs

- Nonlinear environment poses some complications to interpretation of shocks:
  - In the linear context, IRFs are derivatives with respect to u_{it}, V_t.
  - Innovations u_{it}, V_t defined by normalizations adopted for convenience, not economics.
  - IRFs/FEVDs as transmission/importance of some shock are subject to ambiguity.

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  - IRFs/FEVDs as transmission/importance of some shock are subject to ambiguity.
- We extend to our macro-micro setup the idea in Gallant, Rossi, Tauchen (1993):
  - Idea. Fix initial state benchmark values, perturb them and track evolution.
- Perturbation: $\kappa(\delta)$ such that $g(x^b) = g(x^b + \kappa(\delta)) - \delta$.
  - Rule g translates change δ to relevant units, comparable across individuals.
    - Unit perturbation: $g(x) = x$.
    - Rank perturbation: $g(x) = F(x)$.
    - Lorenz-curve perturbation: $g(x) = \left(\int_{-\infty}^{\infty} \xi f(\xi)\, d\xi\right)^{-1} \int_{-\infty}^{x} \xi f(\xi)\, d\xi$.

Nonlinear IRFs: defining innovations

- Macro impulse response of η:
  - For benchmark $Z_1 = Z^b$ and perturbation $g(Z^b) = g(Z^b + \kappa(\delta)) - \delta$:

$$\overline{\text{IRF}}_{it}(Z^b, \eta_{i0}, Z_0) = \lim_{\delta \to 0} \frac{E\left[\eta_{it} \middle| Z_{i1} = Z^b + \kappa(\delta), \eta_{i0}, Z_0\right] - E\left[\eta_{it} \middle| Z_{i1} = Z^b, \eta_{i0}, Z_0\right]}{\delta}.$$

  - Implied macro shock:

$$\tilde{V}_1 = g\left(Q_Z(Z_1|Z_0)\right) - g(Z^b)$$

    which reduces to $\tilde{V}_1 = V_1$ when g is the unit perturbation rule.

- IRF $=$ derivative of expectation with respect to implied innovation (only works locally).

Nonlinear IRFs: defining innovations

- Micro impulse response of η:
  - For benchmark $\eta_{i1} = \eta^b$ and perturbation $g_i(\eta^b) = g_i(\eta^b + \kappa_i(\delta)) - \delta$:

$$\text{IRF}_{it}(\eta^b, Z_1, Z_0) = \lim_{\delta \to 0} \frac{E\left[\eta_{it} \middle| \eta_{i1} = \eta^b + \kappa_i(\delta), Z_1, Z_0\right] - E\left[\eta_{it} \middle| \eta_{i1} = \eta^b, Z_1, Z_0\right]}{\delta}.$$

  - Implied micro shock:

$$\tilde{u}_{i1} = g_i\left(Q_\eta(\eta_{i1}|\eta_{i0}, Z_1, Z_0)\right) - g_i(\eta^b)$$

    which reduces to $\tilde{u}_{i1} = u_{i1}$ when g is the rank perturbation rule.

- IRF = derivative of expectation with respect to implied innovation (only works locally).

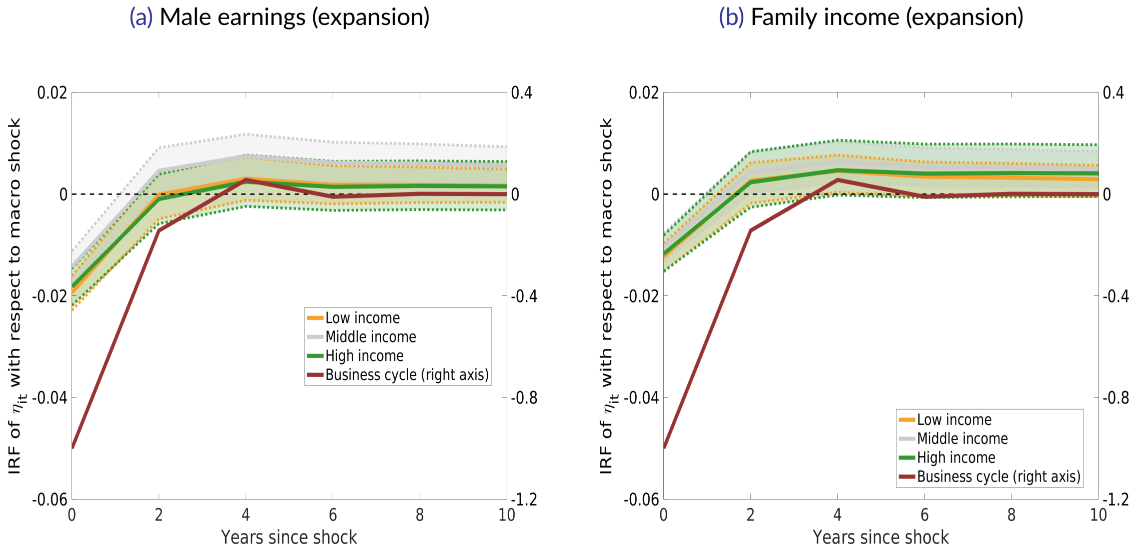
Nonlinear IRFs: discussion

- This is useful for interpretation:
  - Innovations $\tilde{u}_{i1}, \tilde{V}_1$ generalize recursive identification (Cholesky) in linear system with macro state ordered first (with conditional independence replacing orthogonality).

- Link to nonlinear persistence:

$$\text{IRF}_{it}(\eta^b, Z_1, Z_0) = E\left[\prod_{s=2}^{t} \rho\left(u_{is}, \eta_{i,s-1}, Z_t, Z_{t-1}\right) \middle| \eta_{i1} = \eta^b, Z_1, Z_0\right] \times \left(g_i'(\eta^b)\right)^{-1}$$

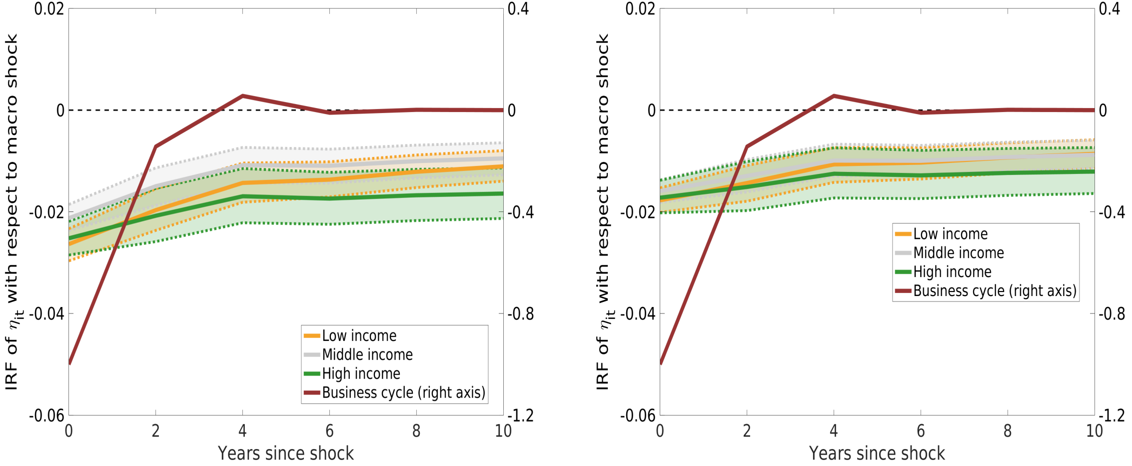
IRFs with respect to macro shocks



IRFs with respect to macro shocks

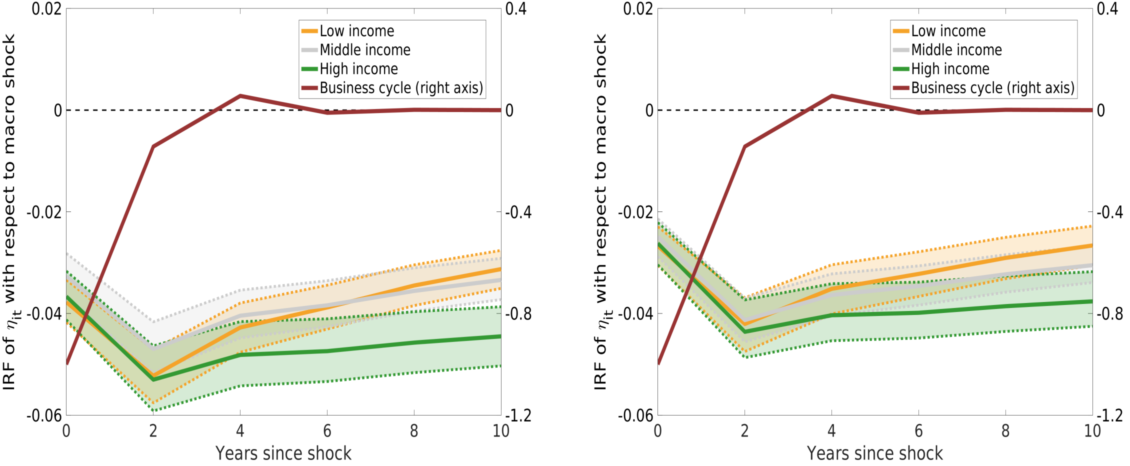
(a) Male earnings (steady state)

(b) Family income (steady state)



IRFs with respect to macro shocks

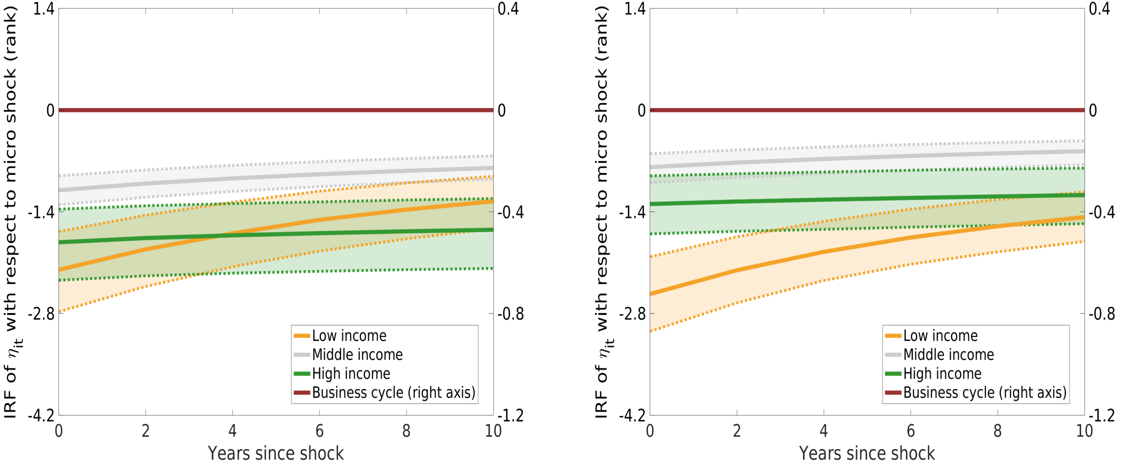
(a) Male earnings (recession) (b) Family income (recession)



IRFs with respect to micro shocks

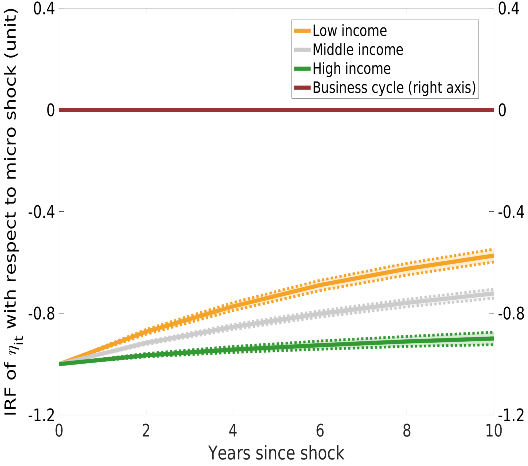
(a) Male earnings (rank rule)

(b) Family income (rank rule)

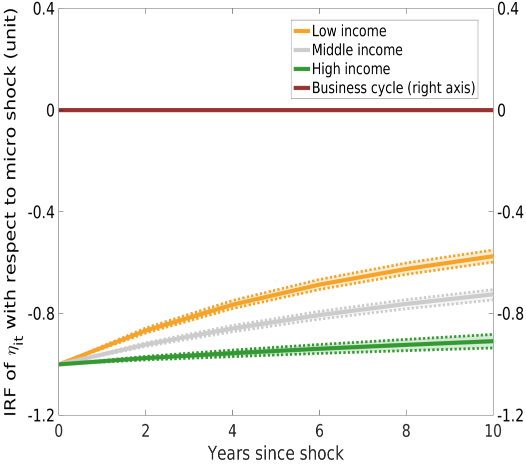


IRFs with respect to micro shocks

(a) Male earnings (unit rule)



(b) Family income (unit rule)



Risk calculations

Cost of business cycles: role of macro nonlinearities

- Very rough macro/micro risk calculation: find CV such that

$$E\left[\sum_{t=1}^{H} \beta^t U\Big((1 - \text{CV}) \exp(\eta_{it})\Big) \Big| \text{no shocks}, \eta_{i0}, Z_0\right] = E\left[\sum_{t=1}^{H} \beta^t U\Big(\exp(\eta_{it})\Big) \Big| \eta_{i0}, Z_0\right],$$

for some utility function $U(\cdot)$.

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  for some utility function $U(\cdot)$.

- Part of the literature focuses on curvature in preferences.
  - E.g., log-utility with exponential income process $\implies$ very little risk.
  - Typically need high risk-aversion to obtain even moderate costs of business cycles.
- Another channel: interaction between marginal utility and macro nonlinearities:
  - Key: presence of quadratic (Z_t, Z_{t-1})-term in Q_η.

Risk calculation: approximations

- For given η_{i0}, Z_0, write $\eta_{it} = \eta_t(\mathbf{u}_i, \mathbf{V})$ with $\mathbf{u}_i, \mathbf{V}$ history of micro/macro shocks.
- Curvature is determined by $\tilde{U}(c) = U(\exp(c))$.
  - CRRA: $U(C) = (C^{1-\zeta} - 1)/(1-\zeta) \implies \tilde{U}'(c) = e^{(1-\zeta)c}$ and $\tilde{U}''(c) = (1-\zeta)e^{(1-\zeta)c}$.
- Compensating variation for macro risk:

$$\text{CV} \approx -\frac{\sum_{t=1}^{H} \beta^t \sum_{\ell=0}^{t-1} \left[\tilde{U}''(\eta_t(\mathbf{0},\mathbf{0})) \left(\frac{\partial \eta_t(\mathbf{0},\mathbf{0})}{\partial V_{t-\ell}}\right)^2 + \tilde{U}'(\eta_t(\mathbf{0},\mathbf{0})) \left(\frac{\partial^2 \eta_t(\mathbf{0},\mathbf{0})}{\partial V_{t-\ell}^2}\right)\right]}{\sum_{t=1}^{H} \beta^t \tilde{U}'(\eta_t(\mathbf{0},\mathbf{0}))}.$$

- Similar approximation holds for micro risk.

Risk calculation: approximations

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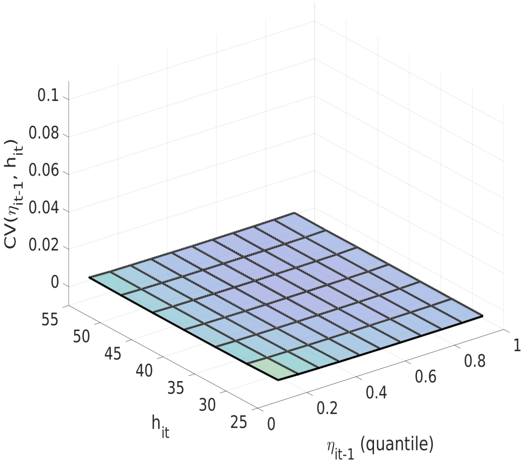
- Curvature is determined by $\tilde{U}(c) = U(\exp(c))$.
    - Log-utility: $U(C) = \ln(C) \implies \tilde{U}'(c) = 1$ and $\tilde{U}''(c) = 0$.
- Compensating variation for macro risk (log-utility):

$$\text{CV} \approx -\frac{\sum_{t=1}^{H} \beta^t \sum_{\ell=0}^{t-1} \left(\frac{\partial^2 \eta_t(\mathbf{0},\mathbf{0})}{\partial V_{t-\ell}^2} \right)}{\sum_{t=1}^{H} \beta^t}.$$

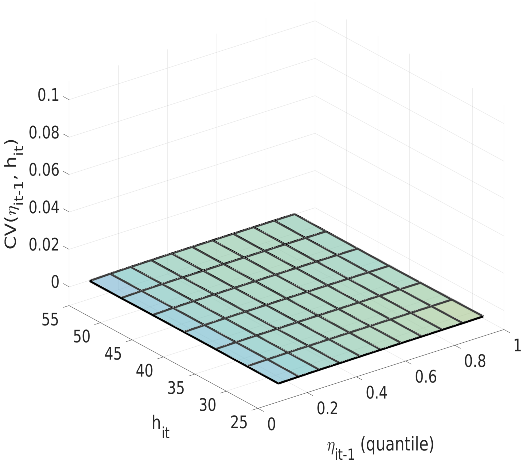
- Similar approximation holds for micro risk.

Macro risk

(a) Male earnings (Q_η linear in Z_t)

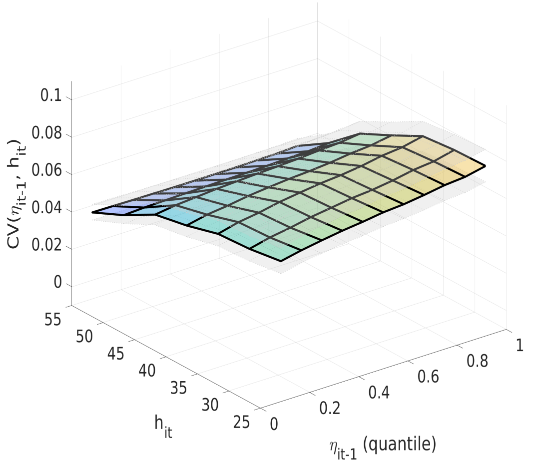


(b) Family income (Q_η linear in Z_t)



Macro risk

(a) Male earnings (Q_η quadratic in Z_t)



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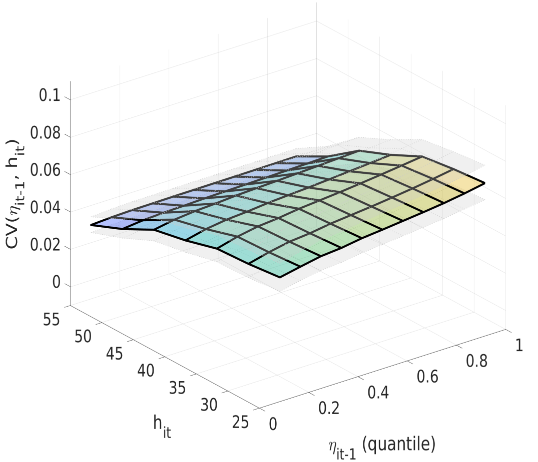
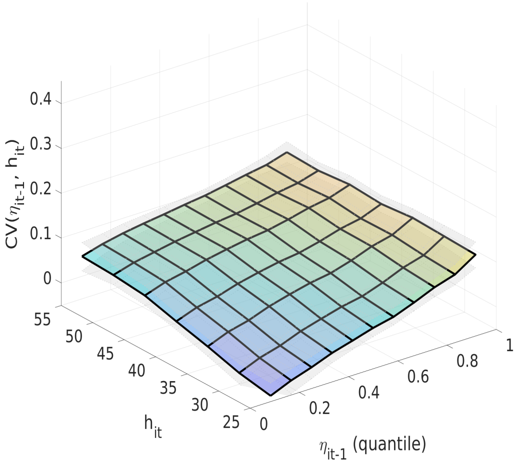
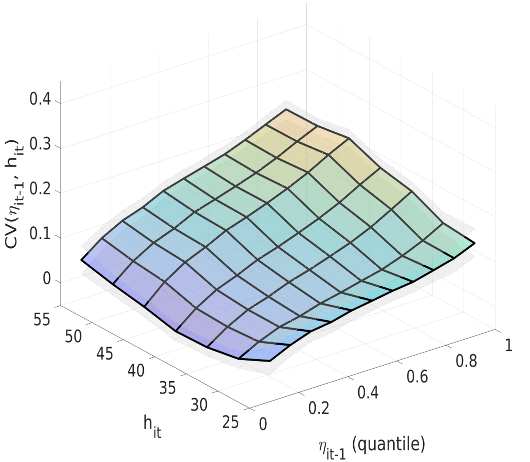


Micro risk

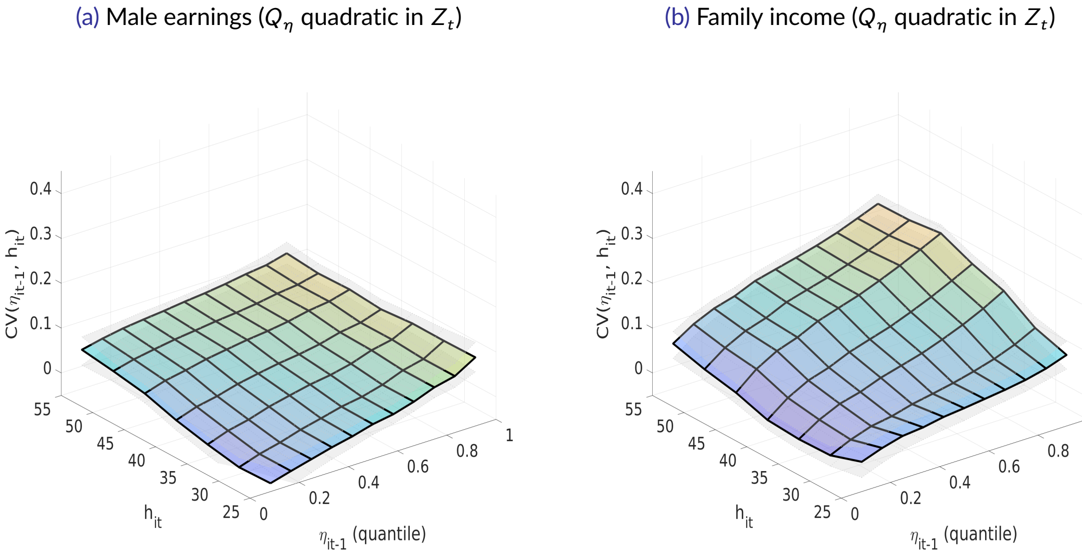
(a) Male earnings (Q_η linear in Z_t)



(b) Family income (Q_η linear in Z_t)



Micro risk


(a) Male earnings (Q_η quadratic in Z_t)
(b) Family income (Q_η quadratic in Z_t)
CV(η_{it-1}, h_{it})
0
0.1
0.2
0.3
0.4
h_{it}
25
30
35
40
45
50
55
η_{it-1} (quantile)
0
0.2
0.4
0.6
0.8
1

Conclusion

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- Building nonlinear reduced forms for heterogeneous agents models with macro shocks:
  - Useful to assess their fit and micro implications.
  - It can help uncover empirical patterns to target in structural approaches.
  - Key goal is to confront the micro data without forcing linearity upon them.
  - We study identification, estimation and inference tools for this purpose.
- Interpretation of impulse responses and shocks is more delicate in a macro/micro setup:
  - Ideally, guide choice of scale of shocks to achieve comparability across individuals.
  - Dynamics can be summarized by measures of nonlinear persistence.
- Nonlinearities in the micro impact of macro shocks matter for welfare calculations:
  - Information about these impacts accumulates slowly (time series rate).
  - One avenue we are exploring: model/estimation uncertainty in the risk calculations.

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Thank you!